# TMA4170 Fourier Analysis

## Fourier transform in $S(\mathbb{R})$:

Schwartz space of rapidly decreasing functions:

$$S(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) : \ \forall k, l \in \mathbb{N}, \ \sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \}$$

Ex: $f(x) = e^{-x^2}$

**Theorem 1:**

$$f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R})$$

$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ is well-defined

**Proposition 1:** Properties

(a) $\mathcal{F}[f(x+h)](\xi) = \hat{f}(\xi) e^{2\pi i h \xi}$

(b) $\mathcal{F}[f(x) e^{-2\pi i x h}](\xi) = \hat{f}(\xi + h)$

(c) $\mathcal{F}[f(\delta x)](\xi) = \delta^{-1} \hat{f}(\delta^{-1} \xi)$

(d) $\mathcal{F}[f'(x)](\xi) = 2\pi i \xi \hat{f}(\xi)$

(e) $\mathcal{F}[(-2\pi i x) f(x)](\xi) = \hat{f}'(\xi)$

# The Gaussian

Theorem 2: $K(x) := e^{-\pi x^2} \implies \hat{K}(\xi) = K(\xi) = e^{-\pi \xi^2}$ K invariant under $\mathcal{F}$

Gaussian: $K_\delta(x) := \frac{1}{\sqrt{\delta}} K\left(\frac{x}{\sqrt{\delta}}\right) \overset{\text{Thm. 2}}{\underset{\text{scaling}}{\implies}} \hat{K}_\delta(\xi) = e^{-\pi \delta \xi^2}$

$K_\delta$ is a good kernel / approximate $\delta$ (Theorem 3):

(i) $\int_{-\infty}^{\infty} K_\delta(x)\, dx = 1$

(ii) $\int_{-\infty}^{\infty} |K_\delta(x)|\, dx \leq M$ (= 1 here)

(iii) $\forall\, \eta > 0$, $\int_{|x|>\eta} |K_\delta(x)|\, dx \longrightarrow 0$ as $\delta \longrightarrow 0$

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## Interchanging integrals in $\mathbb{R}^2$

$$f \in C_m(\mathbb{R}^2) \implies \iint_{\mathbb{R}} f(x_1,x_2)\,dx_1\,dx_2 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x_1,x_2)\,dx_2 \right) dx_1$$

## Convolution in $\mathbb{R}$

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dy$$

well-defined if $f \in C_m(\mathbb{R})$ (or $L^1(\mathbb{R})$)

## Good kernels in $\mathbb{R}$

Good kernels $\{G_\delta\}_{\delta>0}$: (i) $\int_{\mathbb{R}} G_\delta = 1$, (ii) $\int_{\mathbb{R}} |G_\delta| \le M$, (iii) $\int_{|y|>\eta} |G_\delta| \xrightarrow[\delta\to 0]{\eta>0} 0$

Theorem:

$$\boxed{f \in C_m(\mathbb{R}) \implies f * G_\delta \xrightarrow[\delta \to 0]{\text{uniformly}} f}$$

# Gaussians and Fourier inversion

Gaussian $K_\delta(x) = \delta^{-\frac{1}{2}} K(\delta^{-\frac{1}{2}} x)$, $K(x) = e^{-\pi x^2}$

(a) Good kernel, $K_\delta * f \xrightarrow[\delta \to 0]{\text{uniformly}} f$ for $f \in C_m(\mathbb{R})$

(b) $K = \hat{K} \implies K_\delta = \widehat{\left(e^{-\pi \delta \xi^2}\right)} = \hat{\hat{K}}_\delta$

## Multiplication formula:

$$f, g \in S(\mathbb{R}) \implies \int_{\mathbb{R}} f(x) \hat{g}(x)\, dx = \int_{\mathbb{R}} \hat{f}(y) g(y)\, dy$$

## Fourier inversion:

$$f \in S(\mathbb{R}) \implies f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi$$